PHYS 477/577

Lecture Note 1

Title: Propagation of an Electromagnetic Wave in an Optical Medium - Part 1

1 Overview

1. Start from Maxwell's equation in vacuum
2. Obtain Maxwell's equation in macroscopic formulation
3. Obtain the EM wave equation
4. Understanding linear and nonlinear polarization

2 Maxwell's equation

2.1 Maxwell's equation in vacuum (integral form)

In elementary physics classes, Maxwell's equation are commonly written in integral forms and initially for vacuum. Using typical notation and SI units, they are:

$$\oint_{\partial\Omega} \mathbf{E}\cdot \mathrm{d}\mathbf{A} = \frac{1}{\epsilon_0}\int_{\Omega} \rho \mathrm{d}V = \frac{Q_{\text{enclosed}}}{\epsilon_0}, \tag{1}$$

$$\oint_{\partial\Omega} \mathbf{B}\cdot \mathrm{d}\mathbf{A} = 0, \tag{2}$$

$$\oint_{\partial\Sigma} \mathbf{E}\cdot \mathrm{d}\mathbf{l} = -\frac{d}{dt}\int_{\Sigma} \mathbf{B}\cdot \mathrm{d}\mathbf{S} = -\frac{d\Phi_B}{dt}, \tag{3}$$

$$\oint_{\partial\Sigma} \mathbf{B}\cdot \mathrm{d}\mathbf{l} = \mu_0\left(\int_{\Sigma} \mathbf{J}\cdot \mathrm{d}\mathbf{S} + \epsilon_0\frac{d}{dt}\int_{\Sigma} \mathbf{E}\cdot \mathrm{d}\mathbf{S}\right) = \mu_0\left(I_{\text{enclosed}} + \epsilon_0\frac{d\Phi_E}{dt}\right), \tag{4}$$

where Ω and $\partial\Omega$ denote a volume and the boundary of that volume, i.e., a closed surface, respectively. Similarly, Σ and $\partial\Sigma$ denote a surface and the boundary of that surface, i.e., a closed loop, respectively. Eq.(1)-(4) are known as Gauss's law for electric field, Gauss's law for magnetic field, Faraday's law and Ampere's law with Maxwell's correction, respectively.

If the medium is not vacuum, then ϵ_0 and μ_0 get replaced with ϵ and μ, i.e., their values change depending on the medium.

Starting from here, it is easy to prove that any wave propagation must have both $\mathbf{E}$ and $\mathbf{B}$ components that oscillate and that the wave goes at the speed of light, $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$ in vacuum or $v = \frac{1}{\sqrt{\mu \epsilon}}$ in a medium.

We also, often, write $v = \frac{c}{n}$, where n is greater than 1 and is called the index of refraction,

$$n = \sqrt{\frac{\epsilon}{\epsilon_0}\frac{\mu}{\mu_0}}. \tag{5}$$

However, we will never deal with magnetic material in this course. For us, $\mu \cong \mu_0$.

2.2 Maxwell's equation in vacuum (differential form)

The integral form of Maxwell's equation is easy to follow but they are not useful in more advanced courses. Here, we prefer their differential form, i.e.,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \tag{6}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{7}$$

$$\nabla \times \mathbf{E} = -\frac{\partial B}{\partial t}, \tag{8}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right). \tag{9}$$

2.3 Maxwell's equation in material (differential form)

While we are at it, we can also generalize them to a medium, or, again, more accurately, macroscopic formulation in differential form and SI unit:

$$\nabla \cdot \mathbf{D} = \rho_\mathrm{f}, \tag{10}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{11}$$

$$\nabla \times \mathbf{E} = -\frac{\partial B}{\partial t}, \tag{12}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_\mathrm{f} + \frac{\partial \mathbf{D}}{\partial t}. \tag{13}$$

2.4 Maxwell's equation in material (integral form)

$$\oint\limits_{\partial\Omega} \mathbf{D} \cdot \mathrm{d}\mathbf{A} = \int\limits_{\Omega} \rho_\mathrm{f} \mathrm{d}V = Q_\mathrm{f}, \tag{14}$$

$$\oint\limits_{\partial\Omega} \mathbf{B} \cdot \mathrm{d}\mathbf{A} = 0, \tag{15}$$

$$\oint_{\partial\Sigma} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\frac{d}{dt} \int_{\Sigma} \mathbf{B} \cdot \mathrm{d}\mathbf{A} = -\frac{d\Phi_B}{dt}, \tag{16}$$

$$\oint_{\partial\Sigma} \mathbf{H} \cdot \mathrm{d}\mathbf{l} = \int_{\Sigma} \mathbf{J}_{\mathrm{f}} \cdot \mathrm{d}\mathbf{A} + \frac{d}{dt} \int_{\Sigma} \mathbf{D} \cdot \mathrm{d}\mathbf{A} = I_{\mathrm{f}} + \frac{d}{dt} \int_{\Sigma} \mathbf{D} \cdot \mathrm{d}\mathbf{A}. \tag{17}$$

Note that in the equations above, free charges or free charge density and free current or free current density show up naturally (the f and b subscript denote free and bound quantity, respectively). However, in this course, we will always take the free current and charges to be absent! We also have bound charges (unless in vacuum) and (sometimes) bound currents, such that:

$$Q = Q_b + Q_f \quad \text{and} \quad I = I_b + I_f.$$

The influence of the bound charges and currents are "hidden" into **displacement field**, **D**, and **magnetizing field**, **H**

2.5 Bound Charges and Currents

Why did we make this split between free and bound? This is, really, an attempt at simplification, because bound charges and currents are really very complicated. Instead of dealing with this complexity, we work with average values as a sufficiently charge scale (much larger than the atomic scale). We express these average values through so-called "constitutive relations". If we take only linear effect into consideration,

$$\mathbf{D} = \epsilon\mathbf{E} \quad \text{and} \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B}.$$

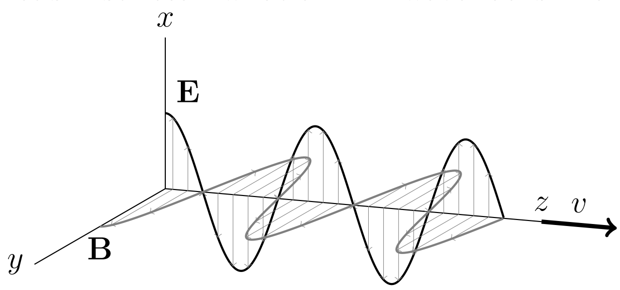
For easy book-keeping, we define a polarization field, **P** and a magnetization field, **M**:

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} \quad \text{and} \quad \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}.$$

However, in this course, we will have $\mathbf{M} = 0$. So, $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B}$. We will also have $\rho_{\mathrm{f}} = 0$ and $\mathbf{J}_{\mathrm{f}} = 0$.

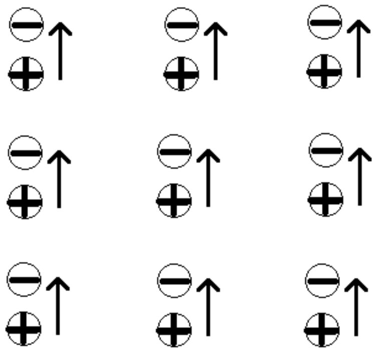
2.6 Why does Polarization exist? What does it mean?

Let's first recall what an EM wave looks like.



When such a wave passes through a material, and remember we do not consider free charges, so all our materials are going to be **dielectrics**, then the oscillating electric field of the wave (laser beam) will cause tiny, oscillating displacement of the electrons (as well as the atomic nuclei, but because they are so much heavier, we ignore that).

Soon, we will consider how this really happens and we will model this behavior with analogy to mechanical oscillators. For now, let's just consider that as these bound charges get displaced and oscillate they will constitute the **dipoles** together with their atomic nuclei.



Polarization or polarization density basically describes the density of their induced dipoles on average as seen in the above picture. Here, each small arrow indicates a tiny "induced" dipoles. We will revisit this in more detail, so let's go back to our equations.

2.7 Maxwell's Equation, as fit for this course

$$\nabla \cdot \mathbf{D} = 0, \tag{18}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{19}$$

$$\nabla \times \mathbf{E} = -\frac{\partial B}{\partial t}, \tag{20}$$

$$\nabla \times \mathbf{B} = \mu_0 \frac{\partial \mathbf{D}}{\partial t}, \tag{21}$$

where we assumed $\rho_\mathrm{f} = 0, \mathbf{J}_\mathrm{f} = 0$ and $\mu = \mu_0$. Here, $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$.

3 Electromagnetic Wave Equation in a dielectric

Now, we can easily determine how $\mathbf{E}$ (alternatively, $\mathbf{B}$) must evolve if it is not constant. Let's take Faraday's law: $\nabla \times \mathbf{E} = -\frac{\partial B}{\partial t}$. Now, taking curl of both sides,

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B} \tag{22}$$

From vector calculus, we know that $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ (the vector Laplacian identity). So, we can write,

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}). \tag{23}$$

Now,since $\nabla \cdot \mathbf{E} = 0$, the gradient of it $\nabla(\nabla \cdot \mathbf{E})$ is also zero. So it follows:

$$-\nabla^2 \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}). \tag{24}$$

Moreover, we can express the curl of the magnetic field in terms of the electric displacement field. Using Ampere's law, $\nabla \times \mathbf{B} = \mu_0 \frac{\partial \mathbf{D}}{\partial t}$, we can write,

$$-\nabla^2 \mathbf{E} = -\frac{\partial}{\partial t}(\mu_0 \frac{\partial \mathbf{D}}{\partial t}) \tag{25}$$

$$\Rightarrow \nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2} \tag{26}$$

But $\epsilon_0 \mu_0 = \frac{1}{c^2}$. So, we get the final form of our wave equation in a dielectric:

$$\boxed{\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2}.} \tag{27}$$

4 Linear and Nonlinear Polarization

Previously, we said we can write $\mathbf{D} = \epsilon \mathbf{E}$ for a medium responding linearly to the wave. In this course, this is not sufficient. We must also consider the nonlinear response. So let's recall the definition of polarization because nonlinearity will arise from the nonlinear polarization response. In more physical terms, nonlinearity will come about because the induced dipoles will depend nonlinearly on $\mathbf{E}$.

We can express the polarization $\mathbf{P}$ as a power series in the electric field strength $\mathbf{E}$ as

$$\mathbf{P} = \epsilon_0 \left(\chi^{(1)}\mathbf{E} + \chi^{(2)}\mathbf{E} + \chi^{(3)}\mathbf{E} + ...\right), \tag{28}$$

where $\chi^{(1)}\mathbf{E}$ is the linear response term and rest of the terms are nonlinear terms. It makes sense to write $\mathbf{P}$ like this, like a Taylor series expansion, because nonlinear effects are normally very weak. We should not take it for granted, in general.

Moreover, the χ terms are tensors in general because they are direction dependent. $\chi^{(j)}$ is called the j-th order electric susceptibility and in general it is a $(j+1)$-th rank tensor. However, in practice we will consider simple polarization, so it acts like a scalar.