PHYS 477/577

Lecture Note 2

Title: Propagation of an Electromagnetic Wave in an Optical Medium - Part 2

1 Overview

1. Slowly varying envelope approximation
2. Brief review of Fourier transform
3. Nonlinear wave equation in frequency domain
4. Simplification of nonlinear wave equation

2 Slowly Varying Envelope Approximation

2.1 Breaking down the electric field into a (slowly varying) envelope and a central (oscillating) frequency

Our main interest is in short pulses, so $\mathbf{E}$ has a temporal structure that is short, but still long compared to an optical cycle. For EM waves in the optical range, say, at a wavelength of $\lambda = 1\mu$m (a very common range used in ultrafast optics), the optical frequency is $f_{\text{opt}} = \frac{c}{\lambda} = 300$THz.

We also commonly define angular frequency as $\omega = 2\pi f$. Then one optical cycle, T_c (one peak from the next peak) is $\frac{1}{f} \approx 3.3$fs for $\lambda = 1\mu$m. The approximation is valid as long as the envelope contains more than a few optical cycles (see Fig. 1 for a typical ultrafast pulse with and without envelope) .

When we talk about ultrafast optics or ultrafast pulses, while there is no "hard limit", it is generally understood that the pulse is < 10ps and most commonly in the femtosecond range ($1\text{ps} = 10^{-12}\text{s}, 1\text{fs} = 10^{-15}\text{s}$). This definition comes from the fact that light mainly interacts with electrons and the characteristic timescale for an excitation of the electrons to transfer to the atoms is in the range of a few picoseconds. Therefore, an ultrashort pulse typically has an **envelope** containing a few to a 100 optical cycles.

Given this overall structure - a superfast oscillating field with a relatively slowly changing (still 'ultrafast) overall amplitude motivates breaking down the electric field expression into an **envelope** and an oscillating field.

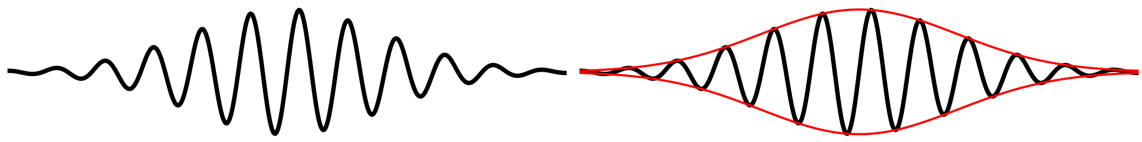

Figure 1: A typical ultrafast pulse in time domain without envelope (left) and with envelope (right).

2.2 Rewriting E(r,t) as an envelope and a (faster) oscillating field

In elementary courses, you would have seen a simpler version numerous times, particularly for solving wave equations. One would assume the general form for a plane wave,

$$\mathbf{E}(\mathbf{r},t) = \mathbf{A}(\mathbf{r},t)\cos(\omega_0 t - k_0 z), \tag{1}$$

where k_0 is the central wavenumber and ω_0 is the central angular frequency. In general, $\mathbf{k}_0 = k_0\hat{\mathbf{z}}$ but we set the coordinates so that the beam goes in the z-direction. You may have seen this earlier where the envelope term was constant.

However, it is much more convenient to work with complex numbers:

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\{\mathbf{A}(\mathbf{r},t)e^{i(k_0 z-\omega_0 t)}\} = \mathbf{A}(\mathbf{r},t)e^{i(k_0 z-\omega_0 t)} + \text{c.c.}, \tag{2}$$

where c.c. denotes the complex conjugate. This is added so that the sum is real valued but we will not always write it explicitly. We will assume that it is there without writing it explicitly.

If $\mathbf{A}$ was constant in time, we could insert this solution into our wave equation (absorbing the $e^{ik_0 z}$ term also into $\mathbf{A}(\mathbf{r})$) and we would obtain the Helmholtz equation. Of course, the Helmholtz equation is for time-independent phenomena. However, our interest is clearly not in case of constant $\mathbf{A}$. To the contrary we are interested in ultrashort pulses, i.e., $\mathbf{A}$ changing rapidly.

3 A Brief Interlude to Fourier transform

In optics, it is most natural to think the quantities in frequency domain. Fourier transform is the mathematical tool we use to achieve that. It allows us to decompose an equation into an infinite sum (i.e., integral) of all different frequencies. The Fourier transform changes time derivatives into $(-i\omega)$'s (see below for proof).

3.1 Nonlinear wave equation in frequency domain

In time domain, we obtained the nonlinear wave equation as:

$$\nabla^2\mathbf{E}(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2\mathbf{E}(\mathbf{r},t)}{\partial t^2} = \mu_0\frac{\partial^2\mathbf{P}(\mathbf{r},t)}{\partial t^2} \tag{3}$$

Expanding $\mathbf{P}$ and separating the linear term from the nonlinear terms:

$$\nabla^2\mathbf{E}(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2\mathbf{E}(\mathbf{r},t)}{\partial t^2} = \mu_0\epsilon_0\chi^{(1)}\mathbf{E} + \text{Nonlinear Terms.} \tag{4}$$

In the following, by $\mathcal{F}\{f(x)\}$ we mean the Fourier Transform of $f(x)$. Now, by definition of Fourier transform,

$$\mathcal{F}\{\mathbf{E}(\mathbf{r},t)\} = \mathbf{E}(\mathbf{r},\omega) = \int_{-\infty}^{\infty} \mathbf{E}(\mathbf{r},t)e^{i\omega t}dt \tag{5}$$

So,

$$\begin{aligned}
\mathcal{F}\{\frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t}\} &= \int_{-\infty}^{\infty} \lim_{\Delta t\to 0} \frac{\mathbf{E}(\mathbf{r},t+\Delta t)-\mathbf{E}(\mathbf{r},t)}{\Delta t} e^{i\omega t}dt \\
&= \lim_{\Delta t\to 0}\left(\int_{-\infty}^{\infty} \frac{\mathbf{E}(\mathbf{r},t+\Delta t)}{\Delta t}e^{i\omega t}dt - \int_{-\infty}^{\infty}\frac{\mathbf{E}(\mathbf{r},t)}{\Delta t}e^{i\omega t}dt\right) \\
&= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(\int_{-\infty}^{\infty}\mathbf{E}(\mathbf{r},t+\Delta t)e^{i\omega(t+\Delta t)}e^{-i\omega\Delta t)}dt - \int_{-\infty}^{\infty}\mathbf{E}(\mathbf{r},t)e^{i\omega t}dt\right) \\
&= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(e^{-i\omega\Delta t)}\mathbf{E}(\mathbf{r},\omega)-\mathbf{E}(\mathbf{r},\omega)\right) \\
&= \mathbf{E}(\mathbf{r},\omega)\lim_{\Delta t\to 0}\left(\frac{e^{-i\omega\Delta t}-1}{\Delta t}\right) \\
&= -i\omega\mathbf{E}(\mathbf{r},\omega).
\end{aligned}$$

That means taking the Fourier transform of a derivative results into the Fourier transform multiplied by $-i\omega$. Therefore, $\mathcal{F}\{\frac{\partial^2\mathbf{E}(\mathbf{r},t)}{\partial t^2}\} = (-i\omega)(-i\omega)\mathbf{E}(\mathbf{r},\omega) = -\omega^2\mathbf{E}(\mathbf{r},\omega)$. Using this above relation, we can write the nonlinear wave equation in frequency domain:

$$\nabla^2\mathbf{E}(\mathbf{r},\omega) + \frac{\omega^2}{c^2}\mathbf{E}(\mathbf{r},\omega) = -\mu_0\epsilon_0\chi^{(1)}(\omega)\omega^2\mathbf{E}(\mathbf{r},\omega) + \mathcal{F}\{\text{Nonlinear Terms}\}. \tag{6}$$

Note that we kept $\chi^{(1)}$ as a function of ω and since there is no general way of executing Fourier transform of nonlinear terms, so we leave it as it is. Later, once we decide to focus on specific nonlinear terms, we will address this issue.

Moreover, most materials have significant dispersion, i.e., $\chi^{(1)}$ (and higher terms) depend on the frequency of the EM wave. This is especially important for us since our focus is ultrashort pulses. By the Fourier theorem, the shorter the pulse, the broader its frequency-converge range, or its spectral width (see Fig. 2).

Let's momentarily forget about the difficulty with taking the Fourier transform of the nonlinear components of the polarization, so we can recapitulate E&M theory and (linear) optics. Using $\mu_0\epsilon_0 = \frac{1}{c^2}$ and after rearranging the nonlinear wave equation in frequency domain, we obtain

$$\nabla^2\mathbf{E}(\mathbf{r},\omega) + \frac{\omega^2}{c^2}\left(1+\chi^{(1)}(\omega)\right)\mathbf{E}(\mathbf{r},\omega) = -\mu_0\omega^2\mathbf{P}_{\mathrm{NL}}(\mathbf{r},\omega), \tag{7}$$

where we kept the nonlinear polarization term in frequency domain but be aware that it's not an exact expression. It's valid for weak nonlinear polarization compared to linear polarization.

Also note that, in general $\chi^{(1)}(\omega)$ is a complex valued function but in a lossless media, the imaginary component is zero. That means if we ignore the imaginary component, we assume that there is no significant loss or gain during propagation. If so, then

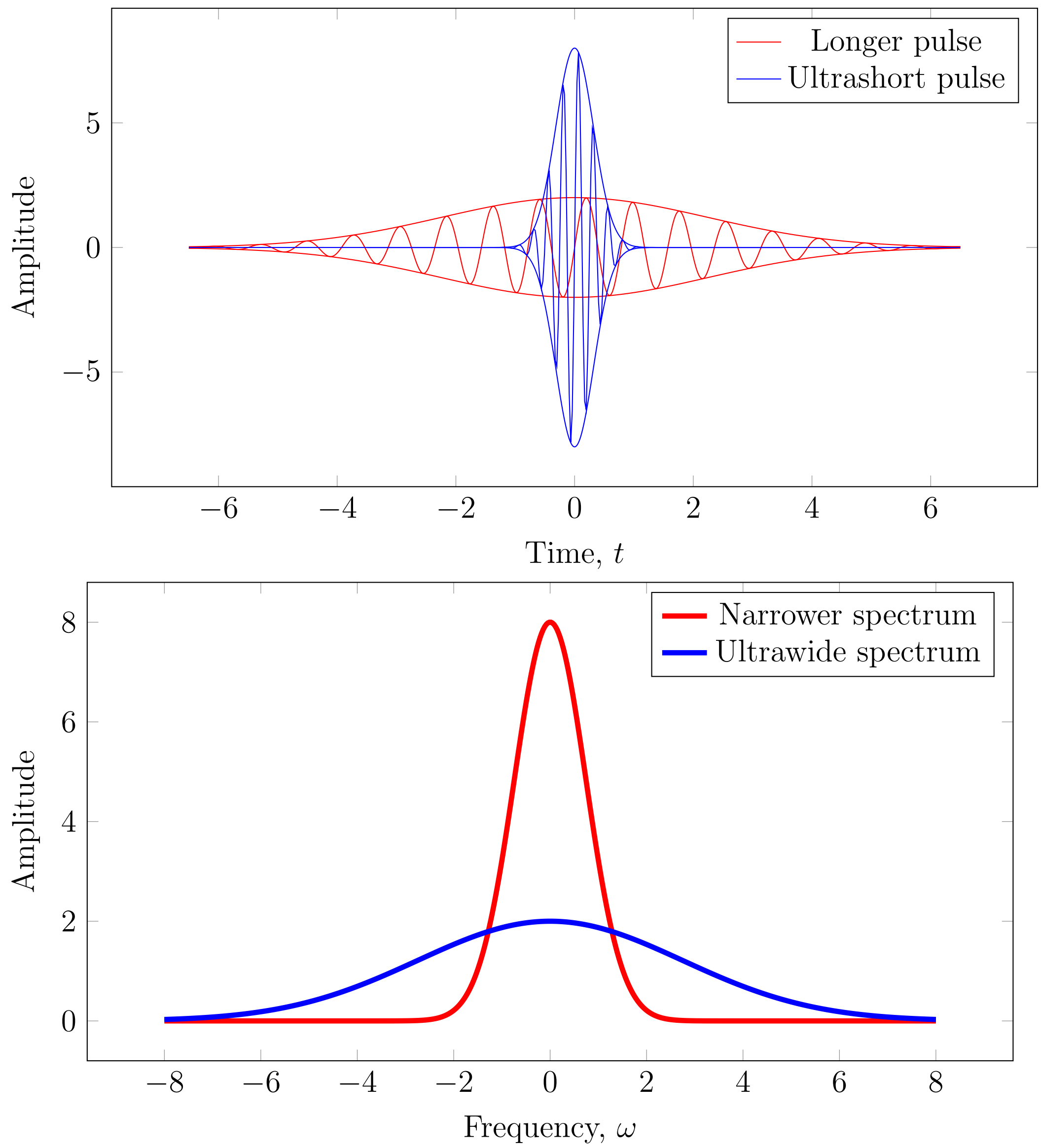

Figure 2: In the top figure two pulses have been depicted in time domain. The blue pulse is an ultrashort pulse that has high amplitude and the red pulse is a longer pulse with relatively low amplitude. In the bottom figure, the same two pulses have been depicted but in frequency domain where we can see that the shorter the pulse in the time domain, the wider its spectral width in frequency domain, and vice-versa.

$1 + \chi^{(1)}(\omega) = n^2(\omega)$, where $n(\omega)$ is the index of refraction and $k(\omega) = \frac{\omega}{c}n(\omega)$ as we know from optics.

3.2 Fourier transform of E(r,t)

By definition,

$$\mathcal{F}\{\mathbf{E}(\mathbf{r},t)\} = \mathbf{E}(\mathbf{r},\omega) = \int_{-\infty}^{\infty} \mathbf{E}(\mathbf{r},t)e^{i\omega t}dt \tag{8}$$

Now using $\mathbf{E}(\mathbf{r},t) = \mathbf{A}(\mathbf{r},t)e^{i(k_0 z-\omega_0 t)}$ (assuming a plane wave),

$$\begin{aligned}\mathbf{E}(\mathbf{r},\omega) &= \int_{-\infty}^{\infty} \mathbf{A}(\mathbf{r},t)e^{i(k_0 z-\omega_0 t)}e^{i\omega t}dt \\ &= \int_{-\infty}^{\infty} \mathbf{A}(\mathbf{r},t)e^{i(\omega-\omega_0)t}dt \cdot e^{ik_0 z} \\ &= \mathbf{A}(\mathbf{r},\omega-\omega_0)\cdot e^{ik_0 z},\end{aligned} \tag{9}$$

where we defined $\mathbf{A}(\mathbf{r},\omega-\omega_0) \equiv \int_{-\infty}^{\infty} \mathbf{A}(\mathbf{r},t)e^{i(\omega-\omega_0)t}dt$

4 Simplification of nonlinear wave equation

From the previous section, we know that

$$\nabla^2\mathbf{E}(\mathbf{r},\omega) + \frac{\omega^2}{c^2}\left(1+\chi^{(1)}(\omega)\right)\mathbf{E}(\mathbf{r},\omega) = -\mu_0\omega^2\mathbf{P}_{\mathrm{NL}}(\mathbf{r},\omega). \tag{10}$$

Also, as remarked earlier, $n(\omega) = \frac{c\cdot k(\omega)}{\omega}$, this leads to $1+\chi^{(1)}(\omega) = \frac{c^2\cdot k^2(\omega)}{\omega^2}$. Let's rewrite the wave equation:

$$\nabla^2\mathbf{E}(\mathbf{r},\omega) + k^2(\omega)\mathbf{E}(\mathbf{r},\omega) = -\mu_0\omega^2\mathbf{P}_{\mathrm{NL}}(\mathbf{r},\omega). \tag{11}$$

4.1 Further Notation Simplification

By focusing on a single polarization direction for the electric field, say, $\mathbf{E} = E\hat{x}$ and $\mathbf{P}_{\mathrm{NL}} = P_{\mathrm{NL}}\hat{x}$, we can further simplify the the wave equation. This way we can drop the vector signs and work with scalar quantities.

Previously we saw that the envelope in the frequency domain is related to the Fourier transform of the electric field:

$$E(x,y,z,\omega) = A(x,y,z,\omega-\omega_0)\cdot e^{ik_0 z}. \tag{12}$$

It is typical for A to have Gaussian shape in x and y; we assume a wave in the plane of (x,y) (perpendicular to z).

The nonlinear polarization term is driven by the electric field, so it will also have a similar form:

$$P_{\mathrm{NL}}(x,y,z,\omega) = P'_{\mathrm{NL}}(x,y,z,\omega-\omega_0)\cdot e^{ik_0 z}. \tag{13}$$

Now inserting all these terms into the nonlinear wave equation eq.11:

$$\nabla^2 A(x,y,z,\omega-\omega_0)\cdot e^{ik_0 z} + k^2(\omega)A(x,y,z,\omega-\omega_0)\cdot e^{ik_0 z} = -\mu_0\omega^2 P'_{\mathrm{NL}}(x,y,z,\omega-\omega_0)\cdot e^{ik_0 z}. \tag{14}$$

Note that here only $e^{ik_0 z}$ term has z-dependency. Now we can express the Laplacian by partial derivative and then evaluate the successive derivative. After that we drop out the common term and we will have:

$$\begin{aligned}&\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)A(x,y,z,\omega-\omega_0) + 2ik_0\frac{\partial A(x,y,z,\omega-\omega_0)}{\partial z} + \\ &\quad k^2(\omega)A(x,y,z,\omega-\omega_0) - k_0^2(\omega)A(x,y,z,\omega-\omega_0) = -\mu_0\omega^2 P'_{\mathrm{NL}}(x,y,z,\omega-\omega_0)\end{aligned} \tag{15}$$

4.2 The wave vector expansion

The wave vector, $k(\omega)$, depends on frequency due to the dispersion property. This dependence varies from material to material. However, to a very good approximation, it varies slowly with ω, which allows us to expand it around ω_0,

$$k(\omega) = k_0 + k_1(\omega - \omega_0) + \frac{1}{2!}k_2(\omega - \omega_0)^2 + \frac{1}{3!}k_3(\omega - \omega_0)^3 + \cdots .$$

Let's connect to regular E&M courses or to linear optics:

- $k_0 = \frac{n_0\omega_0}{c}$ is related to the phase velocity of light
- $k_1 = \frac{\mathrm{d}k}{\mathrm{d}\omega}\big|_{\omega_0} = \frac{1}{v_g}$ is the inverse group velocity.
- $k_2 = \frac{\mathrm{d}^2k}{\mathrm{d}\omega^2}\Big|_{\omega_0}$ is the second order dispersion, aka group velocity dispersion.
- Similarly, $k_n = \frac{\mathrm{d}^nk}{\mathrm{d}\omega^n}\big|_{\omega_0}$, is the n-th order dispersion.

Now we write the wave equation (expressed without the arguments to unclutter the notation) and insert the expansion of $k(\omega)$ into it:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + 2ik_0\frac{\partial}{\partial z} - k_0^2\right) A + k^2 A = -\mu_0\omega^2 P'_{\mathrm{NL}}$$

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + 2ik_0\frac{\partial}{\partial z} - k_0^2\right) A + \left(k_0 + k_1(\omega - \omega_0) + \frac{k_2}{2}(\omega - \omega_0)^2 + \frac{k_3}{6}(\omega - \omega_0)^3 + \cdots\right)^2 A$$
$$= -\mu_0\omega^2 P'_{\mathrm{NL}} \quad (16)$$

4.3 Going back to the time domain

Now we want to go back to the time domain through an inverse Fourier transform. This is easy except that we have many terms with $(\omega - \omega_0)^i$ dependence and one term with ω^2 dependence.

Let's introduce a new variable $\omega' \equiv \omega - \omega_0$ and take the inverse Fourier transform over ω'. We only need to replace $\omega = \omega' + \omega_0 \Rightarrow \omega^2 = (\omega' + \omega_0)^2$. Now taking the inverse Fourier transform of ω^2:

$$\begin{aligned}\omega^2 = (\omega' + \omega_0)^2 &= \omega'^2 + \omega_0^2 + 2\omega'\omega_0 \\ &\xrightarrow{\mathcal{F}^{-1}} -\frac{\partial^2}{\partial t^2} + \omega_0^2 + 2\omega_0 i\frac{\partial}{\partial t} \qquad \text{(because } \omega_0 \text{ is constant)} \\ &= \left(\omega_0 + i\frac{\partial}{\partial t}\right)^2\end{aligned}$$

Note that we will express the amplitude in time domain as $a(t)$ whereas in Fourier domain we express it using $A(\omega)$ to avoid confusion ($a(t)$ or $A(\omega)$ are also functions of x,y,z). Then in time domain our equation becomes,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + 2ik_0\frac{\partial}{\partial z} - k_0^2\right) a(t) + \left(k_0 + ik_1\frac{\partial}{\partial t} + D\right)^2 a(t) = -\mu_0\left(\omega_0 + i\frac{\partial}{\partial t}\right)^2 P_{\mathrm{NL}}(t), \quad (17)$$

where $P_{\mathrm{NL}}(t)$ is the inverse Fourier transform of $P_{\mathrm{NL}}(\omega)$ and we collected all the dispersion terms in D, i.e., $D \equiv -\frac{k_2}{2}\frac{\partial^2}{\partial t^2} + \cdots$.

Throughout our course, we will be going back and forth between time domain and frequency domain many times because some things are better understood in frequency domain such as dispersion, while other things are easier to analyze in time domain such as the nonlinear polarization term. When we will write code for simulation, we will experience this.

Our equation has gotten as complicated as it will get. It is time to start simplifying. First, we will make a reference frame transformation to one that **moves** with the center of the pulse, i.e., at the group velocity. So, the transformation is from (z,t) to (z,τ) where $\tau = t - \frac{z}{v_g}$, which will be centered at the pulse, so it will tell us the time difference with respect to the center of the moving reference frame.

By the chain rule,

$$\frac{\partial}{\partial z} \longrightarrow \frac{\partial}{\partial z}\cdot\underbrace{\frac{\partial z}{\partial z}}_{1} + \frac{\partial}{\partial \tau}\cdot\underbrace{\frac{\partial \tau}{\partial z}}_{\frac{-1}{v_g}=-k_1} = \frac{\partial}{\partial z} - k_1\frac{\partial}{\partial \tau},$$

$$\frac{\partial}{\partial t} \longrightarrow \frac{\partial}{\partial z}\cdot\underbrace{\frac{\partial z}{\partial t}}_{0} + \frac{\partial}{\partial \tau}\cdot\underbrace{\frac{\partial \tau}{\partial t}}_{1} = \frac{\partial}{\partial \tau},$$

$$\begin{aligned}
\text{and } \frac{\partial^2}{\partial z^2} &\longrightarrow \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z} - k_1\frac{\partial}{\partial \tau}\right)\\
&= \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}\right) - \frac{\partial}{\partial z}\left(k_1\frac{\partial}{\partial \tau}\right)\\
&= \left(\frac{\partial}{\partial z} - k_1\frac{\partial}{\partial \tau}\right)\left(\frac{\partial}{\partial z}\right) - \left(\frac{\partial}{\partial z} - k_1\frac{\partial}{\partial \tau}\right)\left(k_1\frac{\partial}{\partial \tau}\right)\\
&= \frac{\partial^2}{\partial z^2} - k_1\frac{\partial^2}{\partial \tau\partial z} - k_1\frac{\partial^2}{\partial \tau\partial z} + k_1^2\frac{\partial^2}{\partial \tau^2}\\
&= \frac{\partial^2}{\partial z^2} - 2k_1\frac{\partial^2}{\partial \tau\partial z} + k_1^2\frac{\partial^2}{\partial \tau^2}.
\end{aligned}$$

Now with the help of these three results, we can write Eq.(17) as:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - 2k_1\frac{\partial^2}{\partial \tau\partial z} + \boxed{k_1^2\frac{\partial^2}{\partial \tau^2}} + i2k_0\left(\frac{\partial}{\partial z} - \boxed{k_1\frac{\partial}{\partial \tau}}\right) - \boxed{k_0^2}\right)a(t) + \left(\boxed{k_0^2} + \boxed{i2k_0k_1\frac{\partial}{\partial \tau}}\right.$$
$$\left. + 2k_0D - \boxed{k_1^2\frac{\partial^2}{\partial \tau^2}} + i2k_1D\frac{\partial}{\partial \tau} + D^2\right)a(t) = -\mu_0\omega_0^2\left(1 + \frac{i}{\omega_0}\frac{\partial}{\partial \tau}\right)^2 P_{\mathrm{NL}}(t). \quad (18)$$

Canceling the color coded terms from Eq.(18) with each other and collecting the similar terms together, we have:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\boxed{\frac{\partial^2}{\partial z^2}}\right)a+i2k_0\left(1-\boxed{\frac{k_1}{k_0}\frac{\partial}{\partial\tau}}\right)\frac{\partial a}{\partial z}+2k_0D\left(1+\boxed{i\frac{k_1}{k_0}\frac{\partial}{\partial\tau}}\right)a+ \boxed{D^2a}=-\mu_0\omega_0^2\left(1+\boxed{\frac{i}{\omega_0}\frac{\partial}{\partial\tau}}\right)^2P_{\mathrm{NL}}. \quad (19)$$

In Eq.(19), all the color coded terms will vanish because of the relative magnitude of these terms. This is the Slowly Varying Envelope Approximation **(SVEA)** that we have been building up to. The reason why these terms can be taken as negligible in comparison with other terms is explained below. Notice that

$$\frac{k_1}{k_0}=\frac{1}{v_g}\frac{c}{n_0\omega_0}=\frac{1}{\omega_0}\frac{c/n_0}{v_g}\cong\frac{1}{\omega_0},\because\ \frac{c/n_0}{v_g}\sim 1\ \text{ and }\ n_0\sim 1.$$

In other words, the $\frac{1}{\omega_0}$ rules the expression which is a very high frequency. Physically, ω_0 is related to the length of one optical cycle which is very very short. Similar reasoning allows us to ignore $\frac{i}{\omega_0}\frac{\partial}{\partial\tau}$ term as well.
Now, consider the term $\frac{k_1}{k_0}\frac{\partial a}{\partial\tau}$. Here, $\frac{\partial a}{\partial\tau}$ is the derivative of the envelope. So, $\frac{k_1}{k_0}\frac{\partial a}{\partial\tau}$ denotes one optical cycle divided by the length of the pulse. So, unless we are dealing with only a few cycle, this term is very small and we can neglect it compared to 1. For instance, this approximation can be applied to a 100fs pulse consists of about 30 cycles at 1μm.

Similarly, $\frac{\partial^2 a}{\partial z^2}\ll k_0\frac{\partial a}{\partial z}$ becomes $\frac{\partial a}{\partial z}\sim\frac{a_{\max}}{d}$ where d is the size of the beam. and $k_0=2\pi/\lambda$. That means unless the beam is focused to the order of $\sim\lambda$, we can ignore this term as well.

Finally, $D^2a\sim\frac{\partial^2 a}{\partial\tau^2}+$ higher order terms. Even $\frac{\partial^3 a}{\partial\tau^3}$ is very weak unless we have very short ($\ll$ 100fs) pulse and at any rate this is the highest-order term we will consider in this class.

In summary, we are left with only

$$i2k_0\frac{\partial a}{\partial z}+\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)a+2k_0Da=-\mu_0\omega_0^2P_{\mathrm{NL}}. \quad (20)$$

Dividing both sides by $i2k_0$, then using the relation $k_0c=\omega_0n_0$ and after than expanding D upto the third order terms:

$$\frac{\partial a}{\partial z}-\underbrace{\frac{i}{2k_0}\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial x^2}\right)a}_{\text{Spatial effects}}+\underbrace{\frac{ik_2}{2}\frac{\partial^2 a}{\partial\tau^2}-\frac{ik_3}{6}\frac{\partial^3 a}{\partial\tau^3}}_{\text{Temporal effects}}=\underbrace{\frac{i\mu_0\omega_0c}{2n_0}P_{\mathrm{NL}}}_{\text{Nonlinear effects}} \quad (21)$$

Due to the coordinate transform, everything in this equation **moves** with the pulse. That is why k_0 and k_1 do not appear. k_0, related to the phase velocity, also does not influence most of the physics we will encounter. So, this equation basically describes how the pulse changes as it propagates.

We are now going to investigate two out of the three effects, namely, temporal (dispersive) effects and nonlinear effects, one by one, before putting them into action together. We will not discuss spatial effects (in detail) until much later in the course. Physically, spatial effects correspond to diffraction and it's only significant when the beam is of a size comparable to the optical wavelength.

Therefore, for the majority of what follows, we will consider:

$$\boxed{\frac{\partial a}{\partial z} + i\frac{k_2}{2} \cdot \frac{\partial^2 a}{\partial \tau^2} - i\frac{k_3}{6} \cdot \frac{\partial^3 a}{\partial \tau^3} = \frac{i\mu_0 \omega_0 c}{2n_0} P_{\mathrm{NL}}} \tag{22}$$